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On the Characterization and Computation of Best Monotone Approximation in $L_p[0, 1]$ for $1 \leqslant p < \infty$

J. J. SWETITS, S. E. WEINSTEIN, AND YUESHENG XU

Department of Mathematics and Statistics, Old Dominion University, Norfolk, Virginia 23508, U.S.A.

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1. INTRODUCTION

For $1 \leqslant p < \infty$, let $L_p = L_p[0, 1]$ denote the Banach space of pth power Lebesgue integrable functions on $[0, 1]$ with $\|f\|_p = (\int_0^1 |f|^p)^{1/p}$. Let $M_p \subset L_p$ denote the closed convex lattice of nondecreasing functions in L_p. Then, $g^* \in M_p$ is called a *best nondecreasing L_p approximation* to $f \in L_p$ if and only if

$$\|f - g^*\|_p \leqslant \|f - g\|_p \qquad \text{for all} \quad g \in M_p.$$

Throughout this paper, whenever ambiguity will not result, g^* may alternatively be called a *best isotone approximation* or simply a *best approximation.*

For $1 \leqslant p < \infty$, each $f \in L_p$ has a best nondecreasng L_p approximation. For $1 < p < \infty$, the best approximation is unique, using the usual convention that any two functions in L_p are equal if they differ on at most a set of Lebesgue measure zero. This convention will be employed throughout this paper.

Constructive solutions to this approximation problem are presented in [3] for $p = 1$, and in [8] for $1 < p < \infty$. The L_∞ case is considered in [9, 10].

In Section 2 of this paper we present an alternative characterization of such best approximations. In Sections 3 and 4 this new characterization is used to develop algorithms for the computation of best monotone L_2 approximation.

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2. Characterization of Best Monotone Approximation

From the duality theory [4], for $1<p<\infty$, $g^* \in M_p$ is the best approximation to $f \in L_p$ if and only if

$$\int_0^1 (g^*-g)\,\phi_{g^*} \geqslant 0 \qquad \text{for all} \quad g \in M_p, \tag{1}$$

where for any $g \in L_p$

$$\phi_g = (f-g)\,|f-g|^{p-2}. \tag{2}$$

For $p=1$, $g^* \in M_1$ is a best approximation to $f \in L_1$ if and only if there exists a $\phi_{g^*} \in L_\infty$ with

$$\|\phi_{g^*}\|_\infty = 1 \tag{3}$$

such that

$$\int_0^1 \phi_{g^*}(f-g^*) = \int_0^1 |f-g^*| \tag{4}$$

and such that (1) holds for all $g \in M_1$.

Because of (3), (4) can be replaced by

$$\phi_{g^*}(x) = \operatorname{sgn}(f(x)-g^*(x)) \qquad \text{whenever} \quad f(x) \neq g^*(x). \tag{4a}$$

Note that for $p=1$, ϕ_{g^*} may not be well-defined by (3) and (4).

This characterization of best isotone approximation depends on the convexity of M_p, and does not utilize the monotonicity of its elements. We shall present an alternative characterization which does utilize the monotonicity, and which is simpler than the above in the sense that unlike (1) which depends on f, g^* and *all* $g \in M_p$, the necessary and sufficient conditions of the new characterization theorem depend solely on f and g^*. Furthermore, several interesting results follow directly from this new characterization, including algorithms for the computation of best approximations.

Definition 1. For $1 \leqslant p < \infty$ let $f \in L_p$, let g^* be a best approximation to f from M_p, and let ϕ_{g^*} be as defined either by (2) for $1<p<\infty$, or by (3) and (4) for $p=1$ (with the noted possible ambiguity). Then define

$$h_{g^*}(x) = \int_0^x \phi_{g^*} \qquad \text{for} \quad x \in [0, 1]. \tag{5}$$

THEOREM 1 (Characterization of Best Nondecreasing Approximation). *For $1 \leqslant p < \infty$, g^* is a best approximation from M_p to $f \in L_p$ if and only if*

$$h_{g^*}(t) \geqslant 0 \qquad \text{for all} \quad t \in [0, 1], \tag{6}$$

$$h_{g^*}(1) = 0, \tag{7}$$

and

$$\text{if } h_{g^*}(t) > 0 \text{ then } g^* \text{ is constant in a neighborhood of } t \in (0, 1). \tag{8}$$

Proof. Smith and Swetits [6] proved the necessity of (6), (7), and (8). We assume these three conditions and show that (1) holds.

For each $g \in M_p$ and each positive integer n define

$$g_n(x) = \begin{cases} g(x), & -n \leqslant g(x) \leqslant n \\ -n, & g(x) < -n \\ n, & n < g(x). \end{cases} \tag{9}$$

Then, pointwise $g_n \to g$, $g_n \phi_{g^*} \to g\phi_{g^*}$, and $|g_n \phi_{g^*}| \leqslant |g\phi_{g^*}|$. By the Lebesgue Dominated Convergence Theorem,

$$\int_0^1 g_n \phi_{g^*} \to \int_0^1 g\phi_{g^*}.$$

Using integration by parts

$$\begin{aligned} \int_0^1 g_n \phi_{g^*} &= -\int_0^1 h_{g^*}\, dg_n && \text{by (7)} \\ &\leqslant 0 && \text{by (6) and since } g_n \text{ is nondecreasing.} \end{aligned}$$

Thus,

$$\int_0^1 g\phi_{g^*} \leqslant 0 \qquad \text{for all} \quad g \in M_p.$$

Similarly, we can define g_n^* as in (9) with g^* replacing g. Then as above $\int_0^1 g_n^* \phi_{g^*} \to \int_0^1 g^* \phi_{g^*}$, and integrating by parts

$$\int_0^1 g_n^* \phi_{g^*} = -\int_0^1 h_{g^*}\, dg_n^* = 0, \qquad \text{by (6) and (8).}$$

Thus,

$$\int_0^1 g^* \phi_{g^*} = 0 \geqslant \int_0^1 g\phi_{g^*} \qquad \text{for all} \quad g \in M_p.$$

Therefore, by (1), g^* is the best approximation to f.

For the case $p=1$, Theorem 1 should be interpreted to mean the following: g^* is a best approximation from M_1 to $f \in L_1$ if and only if *there exist a* ϕ_{g^*} satisfying (3) and (4) and corresponding h_{g^*} satisfying (6), (7), and (8).

In each of the following corollaries to Theorem 1 assume that $1 \leqslant p < \infty$, $f \in L_p$, and g^* is a best nondecreasing L_p approximation to f on $[0, 1]$.

In general g^* is not also a best approximation to f on an arbitrary subinterval of $[0, 1]$. However, we have the following:

COROLLARY 1. (a) *Let* $\alpha \in (0, 1)$ *such that* g^* *is not constant on any neighborhood of* α. *Then,* g^* *is also a best approximation to* f *on both* $[0, \alpha]$ *and* $[\alpha, 1]$.

(b) *Let* $0 < \alpha < \beta < 1$, *such that* g^* *is not constant on any neighborhood of* α, *and is also not constant on any neighborhood of* β. *Then* g^* *is also a best approximation to* f *on* $[\alpha, \beta]$.

Proof. (a) To show that g^* is a best approximation on $[0, \alpha]$ we will show that conditions (6), (7), and (8) of Theorem 1 hold with the interval $[0, 1]$ replaced by $[0, \alpha]$.

Since g^* is best on $[0, 1]$, (6) and (8) imply the corresponding conditions on $[0, \alpha]$. Furthermore, since g^* is not constant on any neighborhood of α, by (6) and (8) we have $h_{g^*}(\alpha) = 0$. Thus g^* satisfies the sufficient conditions that it be best on $[0, \alpha]$.

Next, to show that g^* is best on $[\alpha, 1]$ we show that the sufficient conditions hold when $[0, 1]$ is replaced by $[\alpha, 1]$ and when $h_g(x)$ is replaced by

$$\hbar_g(x) = \int_\alpha^x \phi_g, \qquad \text{for} \quad x \in [\alpha, 1]. \tag{10}$$

However, since

$$h_{g^*}(\alpha) = \int_0^\alpha \phi_{g^*} = 0,$$

we have $\hbar_{g^*}(x) = h_{g^*}(x)$ for $x \in [\alpha, 1]$.

Thus (6), (7), and (8) imply the corresponding conditions for the interval $[\alpha, 1]$, and hence, g^* is also best on $[\alpha, 1]$.

The proof of (b) is similar to (a) and thus is omitted.

COROLLARY 2. *If* g^* *is strictly increasing throughout some interval* $(\alpha, \beta) \subset [0, 1]$, *then* $g^* \equiv f$ *on* (α, β).

Proof. By (8), $h_{g^*}(x)=0$ for all $x \in (\alpha, \beta)$, and therefore,

$$\frac{d}{dx} h_{g^*}(x) = \phi_{g^*}(x) = 0 \qquad \text{for all} \quad x \in (\alpha, \beta).$$

Hence, $g^* \equiv f$ on (α, β).

COROLLARY 3. *If f is nonincreasing on $[\alpha, \beta] \subseteq [0, 1]$, then g^* is constant on $[\alpha, \beta]$.*

Proof. By Corollary 2, g^* is not strictly increasing on any subinterval $(a, b) \subset [\alpha, \beta]$. Thus either g^* has a discontinuity in (α, β), or g^* is constant on $[\alpha, \beta]$.

Suppose that g^* has a discontinuity somewhere in $[\alpha, \beta]$. Then, there exists a $t \in (\alpha, \beta)$ such that

$$g^*(x) > g^*(y) \qquad \text{for all} \quad x \in (t, \beta] \text{ and all } y \in [\alpha, t).$$

Thus by (8), $h_{g^*}(t)=0$, and for any $s \in (\alpha, t)$, $\int_s^t \phi_{g^*} \leqslant 0$, by (6). Therefore, for some $\bar{y} \in (s, t) \subset [\alpha, t)$, $\phi_{g^*}(\bar{y}) \leqslant 0$, which implies that $f(\bar{y}) \leqslant g^*(\bar{y})$. But, since f is nonincreasing on $[\alpha, \beta]$, we have for all $x \in (t, \beta]$ that

$$g^*(x) > g^*(\bar{y}) \geqslant f(\bar{y}) \geqslant f(x).$$

Thus, for all $x \in (t, \beta]$, $\phi_{g^*}(x) < 0$, and

$$h_{g^*}(x) = \int_0^x \phi_{g^*} = \int_t^x \phi_{g^*} < 0 \qquad \text{(contra. (6))}.$$

3. COMPUTATIONAL PRELIMINARIES

Throughout the remainder of this paper assume that $f \in L_2$, and let the terms "least-squares approximation" or "best approximation" signify the best nondecreasing L_2 approximation.

In this section we consider least-squares approximation in three simple cases, which are the building blocks for least-squares approximation to a piecewise monotone function, considered in Section 4.

Case 1. Assume that f is monotone on $[0, 1]$.

If f is nondecreasing on $[0, 1]$, then $g^* = f$ is clearly its best approximation. If f is nonincreasing on $[0, 1]$ then by Corollary 3 its best approximation g^* is a constant K^* on $[0, 1]$, and by (7) of Theorem 1,

$$K^* = \int_0^1 f(x)\, dx.$$

Case 2. Assume that f is nondecreasing on $[0, \alpha]$, and is nondecreasing on $(\alpha, 1]$, for some $\alpha \in (0, 1)$.

Let $M = \text{ess sup}_{x \in [0,\alpha]} f(x)$, and $m = \text{ess inf}_{x \in (\alpha, 1]} f(x)$.

If $M \leqslant m$ then f is nondecreasing a.e. on $[0, 1]$ and this case reduces to Case 1. Thus we assume that $M > m$.

Next, let

$$H(K) = \int_{\substack{0\\ f>K}}^{\alpha} f - K + \int_{\substack{\alpha\\ f<K}}^{1} f - K. \tag{11}$$

We shall show that $H(K)$ has a unique zero K^*, and thus we can define a nondecreasing approximation to f on $[0, 1]$ by

$$g_{K^*}(x) = \begin{cases} K^*, & \text{for } x \in [0, \alpha] \text{ such that } f(x) > K^* \\ & \text{or for } x \in (\alpha, 1] \text{ such that } f(x) < K^* \\ f(x), & \text{elsewhere in } [0, 1]. \end{cases} \tag{12}$$

Furthermore, we shall show that g_{K^*} is the best approximation to f on $[0, 1]$.

LEMMA 1. *$H(K)$ is a strictly decreasing function of K.*

Proof. Let $H_1(K) = \int_{0, f>K}^{\alpha} f - K$. If $K_1 < K_2 < M$, then

$$H_1(K_1) = \int_{\substack{0\\ f>K_1}}^{\alpha} f - K_1 > \int_{\substack{0\\ f>K_1}}^{\alpha} f - K_2 \geqslant \int_{\substack{0\\ f>K_2}}^{\alpha} f - K_2 = H_1(K_2),$$

since $\{x \in [0, \alpha] : f(x) > K_1\} \supseteq \{x \in [0, \alpha] : f(x) > K_2\}$, and since $K_1 < M$ implies that $\{x \in [0, \alpha] : f(x) > K_1\}$ has positive measure.

Thus, H_1 is strictly decreasing for $K < M$.

Furthermore, $H_1(K) = 0$ for all $K \geqslant M$.

Similarly, if we let $H_2(K) = \int_{\alpha, f<K}^{1} f - K$, then we can show that H_2 is strictly decreasing for $K > m$, and $H_2(K) = 0$ for all $K \leqslant m$.

Thus, $H(K) = H_1(K) + H_2(K)$ is strictly decreasing in K.

LEMMA 2. *$H(K)$ is continuous in K.*

Proof. As in Lemma 1 let $H(K) = H_1(K) + H_2(K)$. It suffices to show that H_1 and H_2 are continuous. Assume that $K_1 < K_2$. Then as in Lemma 1, $H_1(K_1) \geqslant H_1(K_2)$. Thus,

$$|H_1(K_1)-H_1(K_2)| = \int_{\substack{0\\ f>K_1}}^{\alpha} f-K_1 - \int_{\substack{0\\ f>K_2}}^{\alpha} f-K_2$$

$$= \int_{\substack{0\\ f>K_2}}^{\alpha} f-K_1 + \int_{\substack{0\\ K_2\geqslant f>K_1}}^{\alpha} f-K_1 - \int_{\substack{0\\ f>K_2}}^{\alpha} f-K_2$$

$$= \int_{\substack{0\\ f>K_2}}^{\alpha} K_2-K_1 + \int_{\substack{0\\ K_2\geqslant f>K_1}}^{\alpha} f-K_1$$

$$\leqslant \int_{\substack{0\\ f>K_2}}^{\alpha} K_2-K_1 + \int_{\substack{0\\ K_2\geqslant f>K_1}}^{\alpha} K_2-K_1 = \int_{\substack{0\\ f>K_1}}^{\alpha} K_2-K_1$$

$$\leqslant \int_0^{\alpha} K_2-K_1 = \alpha(K_2-K_1) < K_2-K_1 = |K_2-K_1|.$$

Thus, H_1 is continuous. Similarly, H_2 is continuous.

LEMMA 3. (a) *$H(K)$ has a unique real zero K^*,* (b) *$m<K^*<M$, and thus g_{K^*} is well-defined by* (12).

Proof. Since $H(K)$ is strictly decreasing it has at most one real zero. Since $H(K)$ is continuous and

$$H(M) = H_1(M) + H_2(M) = H_2(M) < H_2(m) = 0$$

and

$$H(m) = H_1(m) + H_2(m) = H_1(m) > H_1(M) = 0,$$

$H(K)$ has a unique zero K^* in (m, M).

THEOREM 2. *Under the above hypothesis, g_{K^*} (as defined by* (12)) *is the least-squares nondecreasing approximation to f on* $[0, 1]$.

Proof. By the definitions of K^* and g_{K^*}

$$\int_0^1 \phi_{g_{K^*}} = \int_0^1 f - g_{K^*} = H(K^*) = 0.$$

Thus, by Theorem 1, it suffices to show that

$$\int_0^t \phi_{g_{K^*}} = \int_0^t f - g_{K^*} \geqslant 0 \qquad \text{for all} \quad t\in(0,1).$$

First suppose that $t \in (0, \alpha]$. By (12), $f(x) \geqslant g_{K^*}(x)$ for all $x \in [0, t]$. Hence (13) follows in this case.

Next suppose that $t \in (\alpha, 1]$. Then since $H(K^*) = 0$

$$\int_0^t \phi_{g_{K^*}} = -\int_t^1 \phi_{g_{K^*}} \geqslant 0.$$

Case 3. Assume that f is nondecreasing on $[0, \alpha]$, and is nonincreasing on $(\alpha, 1]$ for some $\alpha \in (0, 1)$.

This case is similar to Case 2. Thus the corresponding proofs are omitted and we state the following:

Let

$$H(K) = \int_{\substack{0 \\ f > K}}^{\alpha} f - K + \int_\alpha^1 f - K. \tag{14}$$

Then $H(K)$ has a unique zero K^*, and thus a nondecreasing approximation g_{K^*} is well-defined by

$$g_{K^*}(x) = \begin{cases} K^*, & \text{for } \quad x \in [0, \alpha] \text{ such that } f(x) > K^*, \\ & \text{or for } x \in (\alpha, 1] \\ f(x), & \text{elsewhere on } [0,1], \end{cases} \tag{15}$$

and is the best approximation to f on $[0, 1]$.

4. Least-Squares Approximation

The following theorem provides the last tool that we require for the computation of the best nondecreasing L_2 approximation to $f \in L_2$. It shows how to replace f by another function $\bar{f} \in L_2$, such that f and $\bar{f}$ both have the same best approximation on $[0, 1]$.

THEOREM 3. *Let $f \in L_2$, and let $[a, b] \subseteq [0, 1]$. Let $g^*_{[a,b]}$ denote the best nondecreasing least-squares approximation to f on $[a, b]$. Define*

$$\bar{f} = \begin{cases} g^*_{[a,b]}, & \textit{on } [a, b] \\ f, & \textit{elsewhere on } [0, 1]. \end{cases} \tag{16}$$

Then, f and $\bar{f}$ both have the same best nondecreasing least-squares approximation on $[0, 1]$.

Proof. Since $g^*_{[a,b]}$ is the best approximation to f on $[a, b]$, Theorem 1 yields

$$\int_a^b f = \int_a^b g^*_{[a,b]}, \tag{17}$$

$$\int_a^x f \geqslant \int_a^x g^*_{[a,b]} \qquad \text{for all} \quad x \in [a, b], \tag{18}$$

and

$$\int_a^x f = \int_a^x g^*_{[a,b]}, \qquad \text{for all} \quad x \in (a, b) \text{ such that } g^*_{[a,b]} \text{ is not constant, in any neighborhood of } x. \tag{19}$$

Now, let $\bar{g}$ denote the (unique) best nondecreasing least-squares approximation to $\bar{f}$ on $[0, 1]$. We shall show that $\bar{g}$ is also the best approximation to f on $[0, 1]$.

For $x \in [0, a]$,

$$\int_0^x f - \bar{g} = \int_0^x \bar{f} - \bar{g},$$

for $x \in [b, 1]$,

$$\int_0^x f - \bar{g} = \int_0^a f - \bar{g} + \int_a^b f - \bar{g} + \int_b^x f - \bar{g}$$
$$= \int_0^a \bar{f} - \bar{g} + \int_a^b g^*_{[a,b]} - \bar{g} + \int_b^x \bar{f} - \bar{g} = \int_0^x \bar{f} - \bar{g},$$

and for $x \in (a, b)$,

$$\int_0^x f - \bar{g} = \int_0^a \bar{f} - \bar{g} + \int_a^x f - \bar{g} \geqslant \int_0^a \bar{f} - \bar{g} + \int_a^x g^*_{[a,b]} - \bar{g} = \int_0^x \bar{f} - \bar{g}.$$

Therefore, since g is the best approximation to f on $[0, 1]$, by Theorem 1 we also have

$$\int_0^x f - \bar{g} \geqslant \int_0^x \bar{f} - \bar{g} \geqslant 0 \qquad \text{for all} \quad x \in [0, 1],$$

$$\int_0^1 f - \bar{g} = \int_0^1 \bar{f} - \bar{g} = 0,$$

and

$$\int_0^x f - \bar{g} = \int_0^x \bar{f} - \bar{g} = 0 \qquad \text{for all } x \in [0, a] \cup [b, 1] \text{ such that } \bar{g} \text{ is not constant in any neighborhood of } x.$$

Next, for $x \in (a, b)$ such that $\bar{g}$ is not constant in any neighborhood of x we know by Corollary 3 that $\bar{f}$ cannot be constant on any neighborhood of x.

Since $\bar{f} = g^*_{[a,b]}$ on $[a, b]$, for any such x

$$\int_a^x f = \int_a^x g^*_{[a,b]} = \int_a^x \bar{f},$$

and thus

$$\int_0^x f - \bar{g} = \int_b^a \bar{f} - \bar{g} + \int_a^x f - \bar{g} = \int_0^a \bar{f} - \bar{g} + \int_a^x \bar{f} - \bar{g} = \int_0^x \bar{f} - \bar{g} = 0.$$

Hence by Theorem 1, $\bar{g}$ is the best approximation to f on $[0, 1]$.

DEFINITION 2. Given $f \in L_2$, and $[a, b] \subseteq [0, 1]$, the corresponding function $\bar{f}$ of the form (16) will be called a *refinement of* f. Furthermore, if $\bar{f}$ is a refinement of f, and $\bar{\bar{f}}$ is a refinement of $\bar{f}$, then $\bar{\bar{f}}$ will also be called a refinement of f.

Note that a refinement of $\bar{f}$ (in the above definition) need not be of the form (16). Thus the second sentence in Definition 2 extends the term "refinement" to a larger class of functions.

We now have the following corollary to Theorem 3.

COROLLARY 4. *Let $f \in L_2$, and let $\{f_i\}$ be any sequence of refinements of f. Then,*

(a) *each f_i and f have the same best nondecreasing L_2 approximation on $[0, 1]$, and*

(b) *if $\lim_i f_i = g^*$ exists in L_2, and is nondecreasing, then g^* is the best nondecreasing L_2 approximation to f on $[0, 1]$.*

Corollary 4 is the basis for the following algorithm:

ALGORITHM. *Let $f \in L_2$, and assume that there exists a partition*

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

such that f is monotone on (a_i, a_{i+1}) for $i = 0, 1, ..., n-1$.

(i) *As in Case* 1 (*in Section* 3) *find* $g^*_{[0,a_1]}$, *the best nondecreasing* L_2 *approximation to* f *on* $[0, a_1]$, *and define*

$$f_1 = \begin{cases} g^*_{[0,a_1]} & \text{on } [0, a_1] \\ f & \text{on } (a_1, 1]. \end{cases}$$

(ii) *For* $i = 1, ..., n-1$, *use the methods of Case* 2 *or* 3 (*in Section* 3) *to find* $g^*_{[0,a_{i+1}]}$, *the best nondecreasing* L_2 *approximation to* f_i (*and to* f) *on* $[0, a_{i+1}]$, *and define*

$$f_{i+1} = \begin{cases} g^*_{[0,a_{i+1}]} & \text{on } [0, a_{i+1}] \\ f & \text{on } (a_{i+1}, 1]. \end{cases}$$

For $i \geqslant 1$, f_i *is nondecreasing on* $[0, a_i]$, *and is monotone on* (a_i, a_{o+1}) (*where* $f_i = f$). *Thus, the methods of Case* 2 *or* 3 *can be applied to find the desired best approximation* $g^*_{[0,a_{i+1}]}$.

Each f_i defined in this algorithm is a refinement of f, and thus by Corollary 4, f_n is the best approximation to f on $[0, 1]$.

Remark. This algorithm can be extended to the approximation of any $f \in L_2$, which is piecewise monotone on a countable partition of $[0, 1]$ in each of the following cases:

(i) Assume that there exists a strictly monotone sequence $\{a_i\}_{i=0}^{\infty}$, where $a_0 = 0$, $a_i < 1$ for all i, and $a_i \uparrow 1$, such that f is monotone on (a_i, a_{i+1}) for $i = 0, 1, ...$ Let f_i be as in the algorithm above. Then $f_i \to g^*$, the least-squares approximation to f on $[0, 1]$.

(ii) Assume that f and $\{a_i\}_{i=0}^{\infty}$ are as in case (i) except that $a_i \uparrow a^* < 1$, and that in addition f is monotone on $(a^*, 1]$. Let f_i be as in the algorithm above, let $g^*_{[a^*,1]}$ be the least-squares approximation to f on $[a^*, 1]$ (found by the methods of Case 1), and let

$$g_i = \begin{cases} f_i & \text{on } [0, a^*] \\ g^*_{[a^*,1]} & \text{on } (a^*, 1]. \end{cases}$$

Then as in case (i), $g_i \to g^*$.

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